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# Direct Sums of Self-Small Mixed Groups

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The class $\mathcal{G}$ of self-small mixed abelian groups of finite rank has recently been the focus of a number of investigations. It is known that, up to quasi-isomorphism, $\mathcal{G}$ is dual to the category of locally free torsion-free abelian groups of finite rank. We utilize the reduced mixed groups in $\mathcal{G}$ as building blocks of a more extensive class $\Sigma\mathcal{G}$, the smallest class containing the reduced groups in $\mathcal{G}$ that is closed under taking infinite direct sums and direct summands. Our central result is the determination of a complete set of isomorphism invariants for the groups in $\Sigma\mathcal{G}$. We supplement this broad classification theorem with an investigation of the fine structure of completely decomposable groups in $\Sigma\mathcal{G}$. 

## 1. CANCELLATION AND THE CLASS $\mathcal{G}$

A ring $R$ has *one in the stable range* if, whenever the equation $r_1 s_1 + r_2 s_2 = 1$ holds in $R$, there exists an element $s \in R$ such that $r_1 + r_2 s$ is a unit of $R$. In [B], Bass proves that if $R$ factored by its Jacobson radical is Artinian, then one is in the stable range of $R$. Plainly then, finite rings and finite dimensional rational algebras have one in the stable range.

All groups considered will be additive abelian groups and all homomorphisms will be $Z$-homomorphisms. For a group $G$ we let $E(G)$ be the ring of endomorphisms and $T = T(G)$, $T_p = T_p(G)$ denote the torsion part and the $p$-primary component of the torsion part, respectively.



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A group $G$ has the *substitution property* (in the class of all abelian groups) if, whenever $C$ is a group with $C = G_1 \oplus H = G_2 \oplus K$, $G_1 \cong G_2 \cong G$, then there exists $G_3 \subset C$ such that $C = G_3 \oplus H = G_3 \oplus K$. In [Wa, Theorem 2.1], Warfield shows that a group $G$ (or more generally an $R$-module $M$) has the substitution property if and only if $E(G)$ has one in the stable range. If $G$ has the substitution property, it follows directly that $G$ has the *cancellation property*: If $G \oplus H \cong G \oplus K$ for groups $H, K$, then $H \cong K$. For a nice discussion of these properties, see the introduction to [G].

The class $\mathcal{G}$ consists of self-small mixed groups $G$ such that $G/T$ is finite rank divisible. Recall that a group $G$ is *self-small* if $\mathrm{Hom}(G, \oplus_{i \in I} G_i) \cong \oplus_{i \in I} \mathrm{Hom}(G, G_i)$ for any indexed set of groups $\{G_i : i \in I\}$ with each $G_i \cong G$. A less compact but more revealing alternate description of the groups in $\mathcal{G}$ is that they are of the form $V \oplus G$, where $V$ is finite rank torsion-free divisible and $G$ is a reduced mixed group with torsion subgroup $T = \oplus T_p$ satisfying: (1) Each $T_p$ is finite; (2) $G$ is (can be embedded as) a pure subgroup of $\prod_p T_p$; (3) $G/T$ is finite rank (necessarily divisible); and (4) if $F$ is a maximal rank free subgroup of $G$, then $F$ projects onto $T_p$ for almost all primes $p$. To see that the above two descriptions of $\mathcal{G}$ coincide, see [AGW]. Note that in the definition $G = T$ is allowed. In this case $F = 0$ and condition (4), together with the stipulation that each $T_p$ be finite, requires that $G$ itself be a finite group. For later use we record the following facts.

LEMMA 1. *Let $G \in \mathcal{G}$ be reduced.* (a) *If $F$ is a maximal rank free subgroup then $G/F = D \oplus B$ where $D$ is divisible and $B$ is finite.* (b) *The factor ring $E(G)/\mathrm{Hom}(G, T)$ is a finite dimensional rational algebra.* (c) *If $H \in \mathcal{G}$ is quasi-isomorphic to $G$, then $H = H_0 \oplus B$, $G = G_0 \oplus C$ with $H_0 \cong G_0$ and $B, C$ finite.*

*Proof.* (a) This is easy to prove directly using (2) and (4) of the alternate description of reduced groups in $\mathcal{G}$. Or see [FoW3, Theorem 4]. (b) The ideal $\mathrm{Hom}(G, T)$ is the kernel of the ring homomorphism sending the endomorphism $\theta \in E(G)$ to the induced endomorphism $\bar{\theta} \in E(G/T)$. Since $G/T$ is finite rank divisible $E(G/T)$ is a finite dimensional rational algebra. It is not hard to show that the image of the map $\theta \to \bar{\theta}$ is closed under scalar multiplication by rationals and hence is a rational subalgebra of $E(G/T)$. (c) Suppose that $f\colon G \to H$, $g\colon H \to G$ are homomorphisms such that $fg = n1_H$, $gf = n1_G$ for some natural number $n$. Let $B = \oplus_{p|n} T_p(H)$, $H_0 = H \cap \prod_{(p,n)=1} T_p(H)$. Then $H = H_0 \oplus B$, $B$ is finite, and $nH_0 = H_0$. Define $G_0, C$ similarly for $G$. The result is now clear. ∎

The category $\mathcal{QG}$ has objects the groups in $\mathcal{G}$ and maps quasi-homomorphisms; that is, $\mathrm{Hom}_{\mathcal{QG}}(G, G') = Q \otimes \mathrm{Hom}_Z(G, G')$. A number of papers have dealt with various properties of the groups in $\mathcal{G}$ and the category $\mathcal{QG}$ (see [A, AGW, AH, AJ, FiW, FoW1–3, GW, VW, Wi]). In [Wi] it was shown that $\mathcal{QG}$ is a Krull–Schmidt category with indecomposables of arbitrary finite rank.

In this section we show that groups in $\mathcal{G}$ have the substitution property. The theorem will follow from the already mentioned properties of groups in $\mathcal{G}$, together with [K, Lemma 8], which we now state so as to apply specifically to our situation.

LEMMA 2. *Let $G$ be a group with torsion subgroup $T$. Suppose that one is in the stable range of the rings $\overline{E} = E(G)/\mathrm{Hom}(G,T)$ and $E(T)$. Then $G$ has the cancellation property.*

THEOREM 3. *Groups in the class $\mathcal{G}$ have the substitution* (*and hence the cancellation*) *property in the class of all abelian groups.*

*Proof.* Let $G \in \mathcal{G}$ and let $T = T(G) = \oplus T_p$. Since each $T_p$ is finite, so is its endomorphism ring $E(T_p)$. Hence, one is in the stable range of each $E(T_p)$ and, consequently, one is in the stable range of $E(T) = \prod E(T_p)$. By Lemma 1(b) the ring $\overline{E}$ is a finite dimensional rational algebra. Hence, one is in the stable range of $\overline{E}$. An application of Lemma 2 completes the proof. ∎

## 2. THE CLASS $\Sigma\mathcal{G}$

Recently, Albrecht has considered arbitrary direct sums of groups, $\oplus G_i$, where each $G_i$ is a reduced group in $\mathcal{G}$. In [A] he proves a strong Azumaya-type result: He shows that a summand of such a $\oplus G_i$ is of the form $\oplus H_j$, where for each $j$ there exists $i = i(j)$ such that $H_j$ is a summand of $G_i$. Albrecht's result indicates that this class of groups, which we denote by $\Sigma\mathcal{G}$, might be a reasonably tractable class of mixed groups. In this section we find a set of invariants for groups in $\Sigma\mathcal{G}$ and consider the existence of groups in $\Sigma\mathcal{G}$ with prescribed invariants. Using our invariants, we show that Kaplansky's test problems are true in the class $\Sigma\mathcal{G}$ and that groups in $\Sigma\mathcal{G}$ satisfy the $n$th root property.

The following facts from [Wi] follow easily from the description of reduced groups in $\mathcal{G}$. Any reduced $G \in \mathcal{G}$ is either finite or is a direct sum of finitely many mixed groups which are indecomposable in $\mathcal{QG}$. (By mixed group we always mean "honestly mixed," that is, neither torsion nor torsion-free.) The mixed indecomposables in $\mathcal{QG}$ are the *essentially indecomposable* groups, those mixed groups $G \in \mathcal{G}$ such that if $G = H \oplus K$ then either $H$ or $K$ is finite. Thus, we can write each group in $\Sigma\mathcal{G}$ in the form $\oplus_{i \in I} G_i$ with each $G_i$ an essentially indecomposable mixed group or a finite cyclic group.

Let $\oplus_{i \in I} G_i$ be a group in $\Sigma\mathcal{G}$. For $I' \subset I$ call $\oplus_{i \in I'} G_i$ a *standard summand* of $\oplus_{i \in I} G_i$. Note that $\oplus_{i \in I'} G_i$ is an object of $\mathcal{G}$ precisely when $I'$ is finite. Write $G \sim H$ to indicate that two groups $G, H \in \mathcal{G}$ are quasi-isomorphic.

THEOREM 4. *Let $\oplus_{i \in I} G_i = \oplus_{j \in J} H_j$ be a group in $\Sigma\mathcal{G}$ written in two ways as a direct sum of finite cyclic groups and essentially indecomposable mixed groups, and let $A$ be an essentially indecomposable mixed group in $\mathcal{G}$. Then the cardinalities of the sets $\{G_i : G_i \sim A\}$ and $\{H_j : H_j \sim A\}$ are equal.*

*Proof.* Let $W = \oplus_{t=1}^{n} G_{i_t}$ be a standard summand of $\oplus_{i \in I} G_i$ and $F$ a maximal rank free subgroup of $W$. Since $F$ is finitely generated, $F$ is contained in a standard summand $V = \oplus_{k=1}^{m} H_{j_k}$ of $\oplus_{j \in J} H_j$. By Lemma 1(a) and the fact that all the $H_j$ are reduced, $W$ is quasi-contained in $V$. Hence $W$ is a quasi-summand of $V$. Because $W, V$ are objects of $\mathcal{Q}\mathcal{G}$, a Krull–Schmidt category, each essentially indecomposable summand $G_{i_t}$ of $W$ can be paired with an essentially indecomposable summand $H_{j_k}$ of $V$ such that $G_{i_t} \sim H_{j_k}$. Similar remarks hold for each standard summand $\oplus_{k=1}^{m} H_{j_k}$ of $\oplus_{j \in J} H_j$. It is now easy to prove that, if either of the quasi-isomorphism classes in the statement of the theorem is finite, then so is the other and they will have equal cardinalities. Henceforth, without loss, we assume that both quasi-isomorphism classes are infinite.

Well order the set $\{G_i : G \sim A\}$ as $\{G_\alpha : \alpha < \gamma\}$ for some ordinal $\gamma$. For each $\alpha$ with $\alpha < \gamma$, we attempt to construct smooth ascending chains of standard summands $G^\alpha, H^\alpha$ of $G$ and $H$, respectively, such that: (1) $G_\alpha \subset G^{\alpha+1}$; (2) $G^{\alpha+1} = G^\alpha + W$, $H^{\alpha+1} = H^\alpha + V$, where $W, V$ are in $\mathcal{G}$; (3) each $G_i \sim A$ which occurs as a summand of $G^\alpha$ is a quasi-summand of $H^\alpha$ and each $H_j \sim A$ which occurs as a summand of $H^\alpha$ is a quasi-summand of $G^{\alpha+1}$; and (4) if $G^{\alpha+1}$ properly contains $G^\alpha$, then $H^{\alpha+1} \setminus H^\alpha$ contains at least one $H_j \sim A$. If this construction is successful, it will follow from (1), (2), and (4) that the cardinality of $\{G_i : G_i \sim A\} = \{G_\alpha : \alpha < \gamma\}$ is less than or equal to the cardinality of $\{H_j : H_j \sim A\}$. If the construction fails at some stage, we will be able to conclude directly that our theorem holds.

Set $G^0 = H^0 = 0$, $G^1 = G_0$, $H^1 = V$, where $V \in \mathcal{G}$ is a standard summand of $\oplus_{j \in J} H_j$ quasi-containing $G_0$. Suppose that $\alpha < \gamma$ and that, for all $\beta < \alpha$, $G^\beta, H^\beta$ have been defined satisfying conditions (1)–(4). If $\alpha$ is a limit ordinal, put $G^\alpha = \sum_{\beta < \alpha} G^\beta$, $H^\alpha = \sum_{\beta < \alpha} H^\beta$. Conditions (1), (2), (4), and the second statement of (3) vacuously extend to the $\alpha$th stage. If $G_i \sim A$ occurs as a summand of $G^\alpha$ then it is a summand of some $G^\beta$,

$\beta < \alpha$, so, by induction, $G_i$ is a quasi-summand of the corresponding $H^{\beta}$. Hence $G_i$ is a quasi-summand of $H^{\alpha}$ and (3) holds.

Say $\alpha = \beta + 1$ with $\beta$ a limit ordinal. (i) If $G_{\beta} \subset G^{\beta}$ set $G^{\alpha} = G^{\beta}$, $H^{\alpha} = H^{\beta}$. In this case, (1), (2), (4), and the first statement of (3) are true for the ordinal $\beta + 1$. If $H_j \sim A$ is a summand of $H^{\beta}$ then, by essentially the same argument as in the previous paragraph, $H_j$ is a quasi-summand of $G^{\beta}$. Hence $H_j$ is a quasi-summand of $G^{\beta+1} = G^{\beta}$. (ii) If $G_{\beta}$ is not contained in $G^{\beta}$, put $G^{\alpha} = G^{\beta} \oplus G_{\beta}$, $H^{\alpha} = H^{\beta} + V$, where $V \in \mathscr{G}$ is a standard summand of $\bigoplus_{j \in J} H_j$ quasi-containing $G_{\beta}$. Plainly, conditions (1), (2), and (3) still hold for $\beta + 1$. Note that if $G_{\beta}$ is not contained in $G^{\beta}$ then $V$ is not contained in $H^{\beta}$. Otherwise, by induction, $V$, and hence $G_{\beta}$, would be a quasi-summand of $G^{\beta}$. Hence, $H^{\alpha} = H^{\beta+1}$ contains at least one new $H_j \sim A$ and (4) holds as well.

Finally, suppose $\alpha = \beta + 2$ for some $\beta$. Say that $H^{\beta+1} = H^{\beta} + V$ for a standard summand $V \in \mathscr{G}$ of $\bigoplus H_j$. Define $G^{\alpha} = G^{\beta+1} + W + G_{\beta+1}$, where $W \in \mathscr{G}$ is a standard summand of $\bigoplus G_i$ quasi-containing $V$. Let $H^{\alpha} = H^{\beta+1} + V' + H_j$, where $V' \in \mathscr{G}$ is a standard summand of $\bigoplus H_j$ with $W + G_{\beta+1}$ quasi-contained in $V'$, and $H_j$ is any $H_j \sim A$ not contained in $H^{\beta+1}$. As in the previous paragraph, it is easy to check that conditions (1)–(3) extend to the $\alpha$th stage. In the event that $G^{\alpha} \neq G^{\beta+1}$ we need to add the summand $H_j$ to ensure condition (4). The only obstacle here is that it could be that $G^{\alpha} \neq G^{\beta+1}$ but that each $H_j \sim A$ is already included in $H^{\beta+1}$. In this case, by (2), almost all $H_j \sim A$ are summands of $H^{\beta}$, and hence, by (3), are quasi-summands of $G^{\beta+1}$. An easy calculation now shows that almost all $G_i \sim A$ must be contained in $G^{\beta+1}$. Then it follows directly that the cardinal number of $\{G_i \sim A\}$ is equal to the cardinal number of $\{H_j \sim A\}$.

At this point we can conclude that, in any event, the cardinal number of $\{G_i \sim A\}$ is less than or equal to the cardinal number of $\{H_j \sim A\}$. By a symmetric argument, the proof is complete. ∎

For $G = \bigoplus_{i \in I} G_i$ and $A$ as above, let $\sigma_A(G)$ be the cardinality of the set $\{G_i : G_i \sim A\}$. We have just shown that each $\sigma_A(G)$ is an invariant of $G$.

LEMMA 5. *Let $H, K \in \Sigma\mathscr{G}$ and suppose that $\sigma_A(H) = \sigma_A(K)$ for each essentially indecomposable $A \in \mathscr{G}$. Then there exist groups $G, S, U \in \Sigma\mathscr{G}$ such that $G = \bigoplus_{i \in I} G_i$ with each $G_i$ essentially indecomposable, $S, U$ are direct sums of cyclics, and $H = G \oplus S$, $K \cong G \oplus U$.*

*Proof.* Write $H = \bigoplus_{i \in I} H_i \oplus S'$, $K = \bigoplus_{j \in J} K_j \oplus U'$ with each $H_i, K_j$ essentially indecomposable and $S', U'$ direct sums of cyclics. By assumption, there is a bijection $\mu$: $I \to J$ such that $H_i \sim K_{\mu(i)}$ for each $i$. If we relabel $K_{\mu(i)} = K_i$ we have, by Lemma 1(c), direct sum decompositions

$H_i = G_i \oplus B_i$, $K_i = G_i' \oplus C_i$ with $G_i \cong G_i'$ and $B_i, C_i$ finite. Let $S = S' \oplus [\oplus B_i]$, $U = U' \oplus [\oplus C_i]$. The result is now clear. ▮

For a cardinal number $\alpha$, let $Z(p^k)^\alpha$ denote the direct sum of $\alpha$ copies of the cyclic group $Z(p^k)$. The proof of the next lemma is elementary.

LEMMA 6. *Let $\{G_i : i \in I\}$ be a collection of essentially indecomposable groups in $\mathscr{G}$ such that each $G_i$ has a $Z(p^k)$ summand. Suppose that $\gamma =$ cardinality $I$ is infinite and that $\gamma$ is greater than or equal to a cardinal number $\alpha$. Then $\oplus_{i \in I} G_i \cong \oplus_{i \in I} G_i \oplus Z(p^k)^\alpha$.*

For a prime $p$ and nonnegative integer $k$, let $f_p^k(X)$ be the $p$–$k$ Ulm invariant of a group $X$. We are now ready for the main result of this section, that the cardinal numbers $\sigma_A(G)$ together with the Ulm invariants are a complete set of isomorphism invariants for groups $G$ in $\Sigma\mathscr{G}$.

THEOREM 7. *Let $H, K \in \Sigma\mathscr{G}$ with $\sigma_A(H) = \sigma_A(K)$ for all essentially indecomposable $A \in \mathscr{G}$ and with $f_p^k(H) = f_p^k(K)$ for all $p, k$. Then $H \cong K$.*

*Proof.* By Lemma 5, we have that $H = G \oplus S$, $K \cong G \oplus U$ for $G = \oplus_{i \in I} G_i$ a direct sum of essentially indecomposable mixed groups in $\mathscr{G}$ and $S, U$ direct sums of finite cyclic groups. If $T = T(G)$, then $f_p^k(T) + f_p^k(S) = f_p^k(H) = f_p^k(K) = f_p^k(T) + f_p^k(U)$ for all $p, k$.

Let $\Omega = \{(p,k) : f_p^k(S) \neq f_p^k(U)\}$. If $(p,k) \in \Omega$ it must be that $f_p^k(T)$ is an infinite cardinal with $f_p^k(T) \geq \max(f_p^k(S), f_p^k(U))$. Note that, since each $T_p(G_i)$ is finite, $f_p^k(T)$ is the same as the cardinality of the set $I_p^k = \{i \in I : G_i$ has a $Z(p^{k+1})$ summand$\}$ whenever $(p,k) \in \Omega$. Then since $\{I_p^k : (p,k) \in \Omega\}$ is a countable collection of infinite subsets of $I$, it is a fact of set theory that one may replace each $I_p^k$ by an equipotent subset such that the members of the new collection $\{I_p^k : (p,k) \in \Omega\}$ are pairwise disjoint. Assuming we have done this, for each $(p,k) \in \Omega$ denote $G_{pk} = \oplus_{i \in I_p^k} G_i$. Then, by Lemma 6, $G_{pk} \cong G_{pk} \oplus Z(p^{k+1})^{\operatorname{card} I_p^k}$. Since the $I_p^k$ are pairwise disjoint we obtain $G = \oplus_{i \in I} G_i \cong G \oplus W$ where $W = \oplus_{(p,k) \in \Omega} Z(p^{k+1})^{\operatorname{card} I_p^k}$. Also observe that $S \oplus W \cong W \cong W \oplus U$ because these direct sums of finite cyclics have the same Ulm invariants. Thus $H = G \oplus S \cong G \oplus (S \oplus W) \cong G \oplus (W \oplus U) \cong (G \oplus W) \oplus U \cong G \oplus U \cong K$. We have $H \cong K$ and the proof is complete. ▮

As an easy consequence of Theorem 7, we show that $\Sigma\mathscr{G}$ satisfies Kaplansky's test problems.

COROLLARY 8. *Let $H, K \in \Sigma\mathscr{G}$ with $H^n \cong K^n$ for some natural number $n$. Then $H \cong K$.*

*Proof.* If $H^n \cong K^n$ then $\sigma_A(H^n) = \sigma_A(K^n)$ for all essentially indecomposable $A \in \mathscr{G}$. Hence $\sigma_A(H) = \sigma_A(K)$ for all $A$. Similarly $f_p^k(H) = f_p^k(K)$ for all primes $p$ and natural numbers $k$. By Theorem 7, $H \cong K$. ▮

In particular, the class $\Sigma\mathcal{G}$ satisfies Kaplansky's first test problem: If $H, K \in \Sigma\mathcal{G}$ with $H \oplus H \cong K \oplus K$, then $H \cong K$. A simple argument employing Theorem 7 shows that Kaplansky's second test problem holds for $\Sigma\mathcal{G}$ as well.

COROLLARY 9. *Let $H, K \in \Sigma\mathcal{G}$ be such that each is isomorphic to a summand of the other. Then $H \cong K$.*

We now consider the existence of groups $G$ in $\Sigma\mathcal{G}$ with prescribed Ulm invariants and values $\sigma_A(G)$ for various essentially indecomposable $A \in \mathcal{G}$. (We refer to the latter as the "$\sigma$-invariants" of $G$.) The central issue is how small the Ulm invariants of $G$ can be in relation to the $\sigma$-invariants of $G$. In the following existence theorem, $\mathbf{P} = \{p_0, p_1, p_2, \ldots\}$ denotes the sequence of rational primes.

THEOREM 10. *Let $H = \bigoplus_{i \in I} H_i$, where each $H_i$ is an essentially indecomposable reduced group in $\mathcal{G}$. Let $u$ be a cardinal-valued function defined on $P \times \omega$. The following statements are equivalent.*

(i) *There exists a group $G$ in $\Sigma\mathcal{G}$ with Ulm invariants $u$, and the same $\sigma$-invariants as $H$.*

(ii) *The set $I$ is the union of an ascending chain $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$, where for each $n < \omega$, the Ulm invariants of the standard summand $\bigoplus_{i \in I_n} H_i$ of $H$ are bounded by $u$ for all primes $p \geq p_n$.*

*Proof.* Assume (i) holds. Then by Theorem 4, $G$ has a direct summand $G' = \bigoplus_{i \in I} G_i$ with $G_i$ quasi-isomorphic to $H_i$ for each $i \in I$. For each $i$, we select the smallest prime $p$ such that $T_q(G_i) \cong T_q(H_i)$ for all primes $q \geq p$. This defines a function $\gamma\colon I \to \mathbf{P}$. Put $I_n = \gamma^{-1}\{p_0, \ldots, p_n\}$ for each $n < \omega$. Clearly, $I$ is the union of the chain $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$. By construction, the Ulm invariants of $\bigoplus_{i \in I_n} G_i$ and $\bigoplus_{i \in I_n} H_i$ agree for all primes $p \geq p_n$. Therefore, the Ulm invariants of $\bigoplus_{i \in I_n} H_i$ are bounded by those of $G'$ (hence of $G$) for all primes $p \geq p_n$. Thus, (ii) is verified.

Conversely, assume (ii) holds. For any prime $p_n$ and group $A \in \mathcal{G}$, we temporarily denote $A_n = A \cap \prod_{p \geq p_n} T_p(A)$, a group in $\mathcal{G}$ quasi-isomorphic to $A$. Define $G_i = (H_i)_0$ for all $i \in I_0$, and $G_i = (H_i)_{n+1}$ for all $i \in I_{n+1} \setminus I_n$ $(n < \omega)$. Clearly, $G' = \bigoplus_{i \in I} G_i$ is a group in $\Sigma\mathcal{G}$ with the same $\sigma$-invariants as $H$. Moreover, for each $n < \omega$, all Ulm invariants of $\Sigma_n = \bigoplus_{i \in I_n} G_i$ are bounded by $u$. (Note that the Ulm invariants of $\Sigma_n$ and $\bigoplus_{i \in I_n} H_i$ agree for all primes $p \geq p_n$, and for $0 \leq k \leq n$, the Ulm invariants of $\Sigma_n$ are bounded by those of $\bigoplus_{i \in I_k} H_i$ at all powers of $p_k$.) Since $I = \bigcup_{n < \omega} I_n$, we conclude that the Ulm invariants of $G'$ are bounded by $u$. We may choose a direct sum $T$ of cyclic $p$-groups so that $G = G' \oplus T$ has Ulm invariants $u$. The proof is now complete, since $G$ is a group in $\Sigma\mathcal{G}$ with the same $\sigma$-invariants as $G'$. ∎

We conclude this section with two simple but important applications of Theorem 10.

EXAMPLE 11. A group in $\Sigma\mathcal{G}$ with countably infinite $\sigma$-invariants can have all its Ulm invariants finite.

To construct such a group, let $H_0$ be the pure subgroup of $\prod_{p \in \mathbf{P}} Z(p)x_p$ generated by $\oplus Z(p)x_p$ and the torsion-free element $(x_p)_{p \in \mathbf{P}}$. Put $H = \oplus_{i \in \omega} H_i$, where each $H_i$ is a copy of $H_0$. Let $u$ be the function defined on $\mathbf{P} \times \omega$ by $u(p_n, 0) = n + 1$ and $u(p_n, k) = 0$ for all $n < \omega$ and $k > 0$. Then it is not hard to see that $u$ satisfies statement (ii) in Theorem 10 with $I_n = \{0, \ldots, n\}$. Hence, by Theorem 10, there exists a group $G \in \Sigma\mathcal{G}$ with finite Ulm invariants $u$, and $\sigma_{H_0}(G) = \sigma_{H_0}(H) = \aleph_0$.

EXAMPLE 12. Let $\{H_i : i \in I\}$ be a collection of essentially indecomposable reduced groups in $\mathcal{G}$ with $I$ uncountable. If a group $G \in \Sigma\mathcal{G}$ has the same $\sigma$-invariants as $\oplus_{i \in I} H_i$ then $G$ has uncountable Ulm invariants at infinitely many primes.

To verify this, observe that statement (ii) in Theorem 10 implies the existence of an uncountable subset $J \subseteq I$ such that the Ulm invariants of $H' = \oplus_{i \in J} H_i$ are bounded by those of $G$ for almost all primes. Let $p \in \mathbf{P}$ be an arbitrary prime. For each $i \in J$, we may select $(q_i, k_i) \in \mathbf{P} \times \omega$ such that $q_i > p$ and $f_{q_i}^{k_i}(H_i) \neq 0$. (Otherwise $H_i \in \mathcal{G}$ would be finite.) This determines a function from $J$ to the countable set $\mathbf{P} \times \omega$. Because of the cardinality of $J$, uncountably many $i \in J$ are sent to a single $(q, k)$ with $q > p$. This implies that $f_q^k(H')$ is uncountable. The desired result for $G$ follows immediately.

## 3. MIXED COMPLETELY DECOMPOSABLES

In this section we consider the completely decomposable groups in $\Sigma\mathcal{G}$, that is, those groups of the form $\oplus A_i$, where each $A_i$ is a reduced mixed group in $\mathcal{G}$ of torsion-free rank one. We call these groups *mixed completely decomposable* to distinguish them from ordinary (torsion-free) completely decomposable groups. We show that the mixed completely decomposables share properties analogous to those of the completely decomposables.

We first note that, in view of Theorem 4, mixed completely decomposables can be decomposed in only one way into direct sums of mixed rank ones, but here we have to be satisfied with uniqueness up to quasi-isomorphism.

The main theorem of this section relates mixed completely decomposables to some already well-studied classes of mixed groups. First we need to recall a construction from [FoW2].

Let $G = \oplus G_i$ be in the class $\Sigma\mathscr{G}$ and let $x \in G$ be an element of infinite order. Then $x \in G_0$, where $G_0 \in \mathscr{G}$ is a standard summand of $G$. Since $G_0 \in \mathscr{G}$, $G_0$ can be regarded as a pure subgroup of $\prod T_p$, where $T_p$ is the $p$-torsion subgroup of $G_0$. As discussed in [FoW2], one can construct a smallest subgroup of $G_0$ (up to quasi-equality) that is a reduced rank one mixed group in $\mathscr{G}$ containing $x$. This subgroup, denoted by $\prec x \succ$, is obtained in the following manner. For each $p$, let $\pi_p$ be the projection of $G_0$ into $T_p$ and let $S_p$ be the cyclic subgroup of $T_p$ generated by $\pi_p(x)$. Take $\prec x \succ$ to be the pure subgroup of $\prod S_p$ generated by $\oplus S_p$ and $x$. Since $\oplus S_p$ is the torsion subgroup of $\prod S_p$, taking this pure subgroup in $\prod S_p$ makes sense. If we regard $\prod S_p \subset \prod T_p$ in the natural way, then $\prec x \succ \subset G_0$ and, using the alternate definition of reduced groups in $\mathscr{G}$, it is easy to see that $\prec x \succ$ is as claimed.

THEOREM 13. *The following conditions are equivalent for a mixed group $G$ in $\Sigma\mathscr{G}$.*

(i) *$G$ is completely decomposable.*

(ii) *$G$ is simply presented.*

(iii) *$G$ has a decomposition basis.*

*Proof.* The implication (ii) $\Rightarrow$ (iii) is clear since any simply presented mixed group has a decomposition basis [Wa, Theorem 12]. To prove (iii) $\Rightarrow$ (i), first suppose $G \in \mathscr{G}$ has decomposition basis $\{x_1, \ldots, x_n\}$. Let $G_i = \prec x_i \succ$ for each $i$. If the sum $\Sigma = G_1 + \cdots + G_n \leq G$ fails to be direct, then there is a prime $p$ and integers $m_i$ such that $m_1 x_{1p} + \cdots + m_n x_{np} = 0$, but not all $m_i x_{ip}$ are zero. This implies $\infty = h_p^{T_p(G)}(m_1 x_{1p} + \cdots + m_n x_{np}) = h_p^G(m_1 x_1 + \cdots + m_n x_n) \neq \min\{h_p^G(m_i x_i)\}$, contrary to the definition of decomposition basis. Therefore, $\Sigma$ is completely decomposable of rank $n$. We have $(\Sigma + T(G))/T(G) \cong \Sigma/T(\Sigma) \cong Q^n \cong G/T(G)$. It follows that $(\Sigma + T(G))/T(G) = G/T(G)$, hence $G = \Sigma + T(G)$. Because $T_p(\Sigma) = \langle x_{1p}, \ldots, x_{np}\rangle = T_p(G)$ for almost all $p$, $T(\Sigma)$ has finite index in $T(G)$. Thus $G/\Sigma = (\Sigma + T(G))/\Sigma \cong T(G)/T(\Sigma)$ is finite, and $G$ is quasi-equal to $\Sigma$. Therefore, $G$ is completely decomposable. Because a direct summand of a group with a decomposition basis has one too [AHR], it now follows that a mixed group in $\Sigma\mathscr{G}$ with a decomposition basis is completely decomposable.

Finally, we prove (i) $\Rightarrow$ (ii). It suffices to prove that a rank one group $G \in \mathscr{G}$ is simply presented. Since $G$ is the direct sum of a finite group $S$

(simply presented) and a group with cyclic $p$-components, we may discard $S$ and assume $T_p(G) \cong Z(p^{n_p})$ $(n_p \geq 0)$ for all primes $p$. Then one can show that $G$ is isomorphic to the simply presented group with generators

$$x, x_p^k, \qquad p \in \mathbf{P}, k \geq 1,$$

subject to the relations

$$p^{n_p}x = p^{n_p+1}x_p^1, \qquad px_p^k = p^2x_p^{k+1}, \qquad k \geq 1.$$

Rather than pursue this further, however, we will be content to show how a result from [HR] can be used to prove that $G$ is simply presented (but without providing any clue as to the actual relations involved in a presentation of $G$). After discarding a finite group, we can assume there exists $x \in G$ so that $x_p$ generates $T_p(G)$ for all $p$. In the statement of [HR, Theorem 12.3], take $L = Zx$ to be the valuated cyclic group with valuation induced by the height valuation in $G$. Then the hypotheses of Theorem 12.3 are met if $f$ is taken identically zero. By that theorem, there is a simply presented, rank one group $A$ containing $L$ such that $h_p^A(p^kx) = h_p^G(p^kx)$ for all $p$ and $k$, and such that the relative Ulm invariants of $L$ in $A$ are identically zero. This condition on the relative Ulm invariants forces $A$ to have the same Ulm invariants as $G$. Now by the isomorphism criterion [Wa, Theorem 1], we conclude that $G$ is isomorphic to the simply presented group $A$, as desired. ∎

With Theorem 13 in place, we now consider the problem of characterizing the completely decomposable groups in $\Sigma G$ within the class of all mixed completely decomposable abelian groups. Actually, we do this for groups of rank one. We use $\infty^*$ to denote the sequence $(\infty, \infty, \ldots)$, and $n^*$ to denote $(0, 1, \ldots, n, \infty, \infty, \ldots)$ for any natural number $n$. Let $\langle x \rangle$ be the cyclic group generated by $x$.

THEOREM 14. *Let $G$ be a reduced countable group of torsion-free rank one. The following statements are equivalent.*

(i) *There is a torsion-free element $y \in G$ whose $p$-height sequences are all contained in $\{\infty^*, 0^*, 1^*, \ldots\}$.*

(ii) *$G$ is the direct sum of a group in $\mathscr{G}$ and a torsion group.*

*Proof.* Assume (i) holds. For each prime $p$, let $n_p^*$ denote the $p$-height sequence of $y$ in $G$. If $n_p^* = \infty^*$ for all but finitely many $p$, then $my$ is contained in the divisible part of $G$ for some nonzero integer $m$. This is impossible because $G$ is reduced and $y$ is torsion-free. Hence, the set $\mathbf{S}$ consisting of the primes $p$ for which $n_p^* \neq \infty^*$ is infinite. Take $A \in \mathscr{G}$ to be the pure subgroup of $\prod_{p \in \mathbf{S}} \langle x_p \rangle$ containing $\oplus \langle x_p \rangle$ and $x = (x_p)_{p \in \mathbf{S}}$, where each generator $x_p$ has order $p^{n_p+1}$. Then, for each prime $p$, the $p$-height sequence of $x$ in $A$ is the same as that of $y$ in $G$ (namely $n_p^*$). For any $p \in \mathbf{S}$, we may select $z$ in the $p$-divisible part of $G$ with $p^{n_p+1}z = p^{n_p+1}y$. Then $y - z$ generates a pure subgroup of $G$ of order $p^{n_p+1}$. It follows that $T(A)$ is (isomorphic to) a direct summand of $T(G)$. Clearly, the countable groups $G$ and $A \oplus [T(G)/T(A)]$ have the same Ulm invariants, and contain torsion-free elements ($y$ and $x$) with equal height matrices. Therefore $G \cong A \oplus [T(G)/T(A)]$ by the isomorphism criterion [Wa, Theorem 1], and (ii) holds. Conversely, suppose $G = A \oplus T$ for some rank one $A \in \mathscr{G}$ and torsion group $T$. Let $x = (x_p) \in A \subset \prod T_p(A)$ be of infinite order. By moving a finite direct summand of $A$ over to $T$ we may assume that, for all $p$, $T_p(A) = \langle x_p \rangle$. Under this assumption, if order $x_p = p^{n_p+1}$, we see immediately that all $p$-height sequences of the torsion-free element $x \in G$ are contained in $\{\infty^*, n_p^* : p \in \mathbf{S}\}$. This completes the proof of Theorem 14. ∎

Our final characterization of rank one groups in $\mathscr{G}$ makes use of neat subgroups. A subgroup $H$ of $G$ is *neat* in $G$ if $pH = H \cap pG$ for all primes $p$. Since any rank one group in $\mathscr{G}$ is the direct sum of a finite group and a group in $\mathscr{G}$ with cyclic $p$-components, we lose little by characterizing groups in $\mathscr{G}$ of the latter kind.

THEOREM 15. *Let $G$ be a reduced, rank one group whose torsion subgroup $T = \bigoplus_{p \in \mathbf{S}} T_p$ has cyclic p-components. The following are equivalent.*

(i) *$G$ is in $\mathscr{G}$.*

(ii) *For each $p \in \mathbf{S}$, $G$ has a neat subgroup isomorphic to $Z$ localized at $S - \{p\}$.*

*Proof.* First assume $G \in \mathscr{G}$. Then we may regard $G$ as a pure subgroup of $\prod_{p \in \mathbf{S}} T_p$, and select a torsion-free element $x \in G$ such that for every $p$, the projection $x_p$ of $x$ onto $T_p$ generates $T_p$. For a fixed prime $p \in \mathbf{S}$, let $y = x - x_p \in G$. Since multiplication by $p$ is an automorphism of the subgroup $G \cap \prod_{q \neq p} T_q$ of $G$, there are unique elements $y_p^1, y_p^2, \ldots$ of this subgroup such that $p^i y_p^i = y$ for $i \geq 1$. Similarly, for each prime $q \notin \mathbf{S}$, there are unique elements $y_q^i$ $(i \geq 1)$ of $G$ such that $q^i y_q^i = y$. Define

$$H = \langle y_p^i, y_q^i : i \geq 1, q \notin \mathbf{S} \rangle \subset G.$$

The reader can verify that $H$ is torsion-free of rank one, and that the heights of $y$ in $H$ satisfy $h_p^H(y) = \infty$, $h_q^H(y) = \infty$ for all $q \notin \mathbf{S}$, and $h_q^H(y) = 0$ for all $q \in \mathbf{S} - \{p\}$. It follows immediately that $H$ is isomorphic to the localization of $Z$ at $\mathbf{S} - \{p\}$. To show that $H$ is neat in $G$, we need only establish $H \cap rG \subseteq rH$ for a given prime $r \in \mathbf{S} - \{p\}$. Since the torsion group $H/\langle y\rangle$ is $r$-torsion-free, this inclusion holds provided $\langle y\rangle \cap rG \subseteq rH$. Now if $my \in \langle y\rangle \cap rG$, then the projection of $my$ onto $T_r$ is $mx_r \in rTr = \langle rx_r\rangle$. Because the order of $x_r$ is a positive power of $r$, we deduce that $r$ divides $m$. Thus $my \in rH$, as desired. Statement (ii) is established.

Conversely, assume $G$ satisfies (ii). Since for every $p \in \mathbf{S}$ the group $G$ has a subgroup isomorphic to $Z$ localized at $\mathbf{S} - \{p\}$, we see that for every prime $p$, $G/T$ has a nonzero element of $p$-height $\infty$. Since $G/T$ is torsion-free of rank one, we deduce $G/T \cong Q$. Now since $G$ is reduced and $G/T$ is divisible, one knows that $G$ may be regarded as a pure subgroup of $\operatorname{Ext}_Z(Q/Z, T)$ which is (isomorphic to) $\prod_{p\in\mathbf{S}} T_p$. For a fixed $p \in \mathbf{S}$, let $H$ be a neat subgroup of $G$ isomorphic to $Z$ localized at $\mathbf{S} - \{p\}$. Choose a nonzero element $h \in H$. Then $h$ has $q$-height 0 in $H$ for almost all primes $q \in S$. Since $H$ is neat in $G$, $h$ has $q$-height 0 in $G$ whenever it does in $H$. We conclude that for almost all primes $q \in \mathbf{S}$, the projection of $h$ onto the cyclic component $T_q$ will have $q$-height 0, and therefore generate $T_q$. Now it follows from [AGW] (or see the discussion preceding Lemma 1) that $G$ is in $\mathcal{G}$. This establishes (i), and completes the proof of Theorem 15. ∎

For our additional results about completely decomposable groups in $\Sigma\mathcal{G}$, we need to recall the main theorem [FoW1]: There is a rank preserving duality $d$ from the category $\mathcal{QG}$ to the category with objects locally free torsion-free finite rank groups and maps quasi-homomorphisms.

If $G \in \Sigma\mathcal{G}$ and $x \in G$ is an element of infinite order, the *dual type of* $x$ is the type of the rank one torsion-free group $d \prec x \succ$. Alternately, if $\pi_p$: $G \to T_p(G)$ is projection and $\pi_p(x)$ has order $p^{k_p}$, the *dual characteristic* of $x$ is the characteristic $(k_p)$. The dual type is then the type of the dual characteristic. (See [FoW2] for details.) A group $G$ in $\Sigma\mathcal{G}$ is called *homogeneous* if all of its elements of infinite order have the same dual type.

We now can give mixed group analogues of Lemma 86.8 and Theorem 86.6 of [F].

LEMMA 16. *Let $G, H$ be groups in $\mathcal{G}$ with $G$ homogeneous mixed completely decomposable and $H \subset G$. Then* (a) *$H$ is a quasi-summand of $G$ and* (b) *$H$ is a summand of $G$ if and only if $H$ is pure in $G$.*

*Proof.* (a) Since $G, H$ are in $\mathcal{G}$, so is $G/H$ [FoW2, Theorem 3]. The $\mathcal{QG}$ exact sequence $0 \to H \to G \to G/H \to 0$ can be dualized to produce a quasi-exact sequence of torsion-free groups $0 \to d(G/H) \to dG \to dH \to 0$. In particular, the map $d(G/H) \to dG$ is a quasi-pure embedding. Moreover, since $G = \oplus_{i=1}^n A_i$ is a homogeneous mixed completely decomposable then $dG \sim \oplus_{i=1}^{n} dA_i$ is a (finite rank) homogeneous completely decomposable group. Thus, the map $d(G/H) \to dG$ quasi-splits. We dualize again to obtain (a). Note here the assumption that $H \in \mathcal{G}$ is playing the role played by purity in the torsion-free setting.

(b) By (a), we have that $nG \subset H \oplus H' \subset G$ for some $H' \subset G$ and positive integer $n$. As in the proof of Lemma 1(c), we can write $G = G_0 \oplus B$, $H = H_0 \oplus C$, $H' = H'_0 \oplus C'$, with $G_0 = H_0 \oplus H'_0$ and $C, C', B$ the (finite) $n$-torsion subgroups of $H, H', G$. Plainly, $C \subset B$ and $H$ is pure in $G$ if and only if $C$ is a summand of $B$. This establishes part (b). ∎

THEOREM 17. *Let $H \subset G$ be groups in $\Sigma\mathcal{G}$ with $G$ homogeneous mixed completely decomposable. Then $H = H_0 \oplus T$, where $H_0$ is a mixed completely decomposable group of the same dual type as the dual type of $G$ and $T$ is a direct sum of cyclics.*

*Proof.* Since $H \in \Sigma\mathcal{G}$, $H = \oplus H_j$ where each $H_j$ is either finite rank essentially indecomposable or a finite cyclic group. Let $H_j$ be an essentially indecomposable summand of $H$. It will be enough to show that $H_j$ is a rank one group of dual type the dual type of $G$. As in the proof of Theorem 4, since $H_j$ is of finite rank $H_j$ is quasi-contained in a standard summand $G_0$ of $G$ with $G_0 \in \mathcal{G}$. The previous lemma allows us to conclude that $H_j$ is a quasi-summand of $G_0$. By Theorem 4, $H_j$ is quasi-isomorphic to a (rank one) summand of $G_0$, and the proof is complete. ∎

We want to point out that, in spite of Theorem 17, subgroups of homogeneous completely decomposable groups in $\Sigma\mathcal{G}$ can be extremely complicated.

EXAMPLE 18. The homogeneous, completely decomposable group $H \in \Sigma\mathcal{G}$ (dual type $(1, 1, 1, \dots)$) in Example 11 contains an indecomposable, torsion-free group $E$ of infinite rank.

Recall $H = \oplus_{i \in \omega} H_i$, where $H_i = H_0 \subset \prod_{p \in \mathbf{P}} Z(p)x_p$ for each $i$. To construct $E$, we first construct subgroups of the groups $H_1, H_2, \dots$ similar to those obtained in the proof of Theorem 15. For $i \geq 1$, choose $y_i \in H_i$ so that $(y_i)_{p_0} = (y_i)_{p_i} = 0$, and $(y_i)_p = x_p$ for all other primes $p$. For $k = 1, 2, 3, \dots$, there are (unique) elements $z_i^k \in H_i \cap \prod_{p \neq p_i} Z(p)x_p$ such that $p_i^k z_i^k = y_i$. For $i \geq 1$, we define subgroups

$$E_i = \langle y_i, z_i^1, z_i^2, \dots \rangle \subset H_i.$$

The reader can verify that $E_i$ is a torsion-free group of rank one, and that the characteristic of $y_i$ in $E_i$ is $(0, \dots, \infty, 0, \dots)$, where $\infty$ occurs at the entry corresponding to $h_{p_i}^{E_i}(y_i)$. Now choose elements $p_0^{-1}(y_1 + y_i) \in H$ for $i \geq 2$. Define

$$E = \left\langle \bigoplus_{i \geq 1} E_i;\ p_0^{-1}(y_1 + y_2), p_0^{-1}(y_1 + y_3), \ldots \right\rangle \subset H.$$

Then $E$ is a torsion-free group of infinite rank, and is indecomposable by [F, Lemma 88.3].

EXAMPLE 19. The rank two homogeneous mixed completely decomposable subgroup $H_1 \oplus H_2$ of the group $H$ of Example 18 contains countably many non-quasi-isomorphic strongly indecomposable torsion-free groups of rank two.

Let $p, q, r$ be distinct primes. Choose $u_1 \in H_1$, $u_2 \in H_2$ with $(u_1)_s = (u_2)_s = 0$, $s = p, q, r$ and $(u_1)_s = (u_2)_s = x_s$ for all other primes $s$. As above, choose $v_i^k$, $i = 1, 2$, $k \geq 1$, with $p^k v_1^k = u_1$, $q^k v_2^k = u_2$. Choose $w^k$, $k \geq 1$, with $r^k w^k = u_1 + u_2$ and let $G(p, q, r)$ be the subgroup of $H_1 \oplus H_2$ generated by $\{v_i^k, w^k\colon i = 1, 2,\ k \geq 1\}$. It is easy to check that the collection of subgroups of the form $G(p, q, r)$ is as claimed.

For a characteristic $\chi$ and a mixed group $G \in \Sigma\mathscr{G}$, let $G^d(\chi)$ be $\Sigma_{x \in W} \prec x \succ$, where $W = \{x \in G,\ d\chi(x) \leq \chi\}$. Here $d\chi(x)$ denotes the dual characteristic of $x$. An epimorphism $f\colon G \to H$, with $G, H$ mixed groups in $\Sigma\mathscr{G}$, is called *d-balanced* if $f\colon G^d(\chi) \to H^d(\chi)$ is epic for all characteristics $\chi$. Our final result recalls the well-known characterization of torsion-free completely decomposable groups.

THEOREM 20. *A group $G \in \Sigma\mathscr{G}$ is d-balanced projective if and only if $G = G_0 \oplus T$ with $G_0$ mixed completely decomposable and $T$ a direct sum of finite cyclics.*

*Proof.* Suppose that $G \in \Sigma\mathscr{G}$ is $d$-balanced projective. To show $G = G_0 \oplus T$ as above, it suffices to show that every mixed essentially indecomposable summand $H$ of $G$ is of rank one. Without loss assume that there is a free subgroup of $H$ that projects onto each $T_p(H)$. Then, if $S$ is the set of elements of infinite order in $H$, the natural map from the mixed completely decomposable group $X = \oplus_{s \in S} \prec s \succ$ to $H$ will be a $d$-balanced epimorphism. Since $H$ is $d$-balanced projective, $H$ is isomorphic to a summand of $X$. By Albrecht's theorem, $H$ is of rank one.

Conversely, let $f\colon G \to H$ be a $d$-balanced epimorphism. For each prime $p$, let $\pi_p, \nu_p$ be the projections of $G, H$ onto their $p$-torsion subgroups. We first claim that if $\langle h \rangle$ is a $p^k$-cyclic summand of $H$, then there is a $p^k$-cyclic summand $\langle g \rangle$ of $G$ such that $f(g) = h$. To prove the claim, take an element $y \in H$ of infinite order, modified if necessary so that $\nu_p(y) = h$. (The group $H$ is required to have elements of infinite order by the definition of a $d$-balanced epimorphism.) By the $d$-balanced property there exists $x \in G$ with the same dual characteristics as $y$ such that $f(x) = y$. In particular, $g = \pi_p(x)$ has order $p^k$. We have that $f(g) = h$ and, hence, that $p^{k-1}g$ must have $p$-height precisely $k - 1$ in $G$. Hence $\langle g \rangle$ can be taken as the desired $p^k$-cyclic summand of $G$. In view of our claim it is easy to check that any finite cyclic, hence any direct sum of finite cyclics, is projective with respect to any sequence $f\colon G \to H$ with $f$ a $d$-balanced epimorphism.

Now let $A \in \mathcal{G}$ be of rank one and let $\theta\colon A \to H$. After lifting $\theta$ on a finite direct summand we can assume that our new $A$ has an element $a = (a_p)$ such that $a_p$ generates $T_p(A)$ for all $p$. If $\theta(a)$ has finite order then $A/\ker\theta$ is a finite group and, by the previous paragraph, $\theta$ can be lifted. Otherwise, $\theta(a)$ will be an element of infinite order such that $\lambda = d\chi[\theta(a)] \le d\chi(a)$. Since $f$ is $d$-balanced we can take $z \in G^d(\lambda)$ with $f(z) = \theta(a)$. Since $\lambda \le d\chi(a)$ the assignment $a \to z$ will induce a homomorphism from $A$ to $G$ lifting $\theta$. Thus any rank one group in $\mathcal{G}$ is $d$-balanced projective and the proof of the theorem is complete. ∎

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